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On the Approximation of Müntz Series by Müntz Polynomials

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The approximation of functions by Müntz polynomials $p_n(x)=\sum_{\nu=0}^{n} a_\nu x^{\lambda_\nu}$, $n\in\mathbb{N}$, is studied. Converse theorems are of special interest. Under certain restrictions on the numbers $\lambda_\nu\in\mathbb{R}$, $\nu\in\mathbb{N}$, $0\leqslant\lambda_0<\lambda_1<\cdots\to\infty$, it is shown that a "good" rate of convergence of the error $\|f-p_n\|_{[0,1]}$, as $n\to\infty$, implies the existence of a series $\hat{f}(z)=\sum_{\nu=0}^{\infty} c_\nu z^{\lambda_\nu}$, $z\in\mathbb{C}_{\log}$, absolutely convergent in a certain circular region $G\subset\mathbb{C}_{\log}$ around the branch point zero whose restriction to the real interval $[0,1]$ coincides with the given function $f\in C[0,1]$. ($\mathbb{C}_{\log}$ denotes the Riemann surface of the logarithm.)

1. INTRODUCTION

Let $C[0,1]$ denote the space of real-valued continuous functions on $[0,1]$ endowed with the uniform norm

$$\|f\|_{[0,1]}:=\max\{|f(x)|: x\in[0,1]\},\qquad f\in C[0,1].$$

Let (λ_ν) be a given sequence of real numbers, $\nu\in\mathbb{N}$,

$$0\leqslant\lambda_0<\lambda_1<\cdots,\qquad \lim_{\nu\to\infty}\lambda_\nu=\infty.$$

Given such a sequence (λ_ν) and a number $n\in\mathbb{N}$, let $\prod_n(\lambda_\nu)$ denote the space of Müntz polynomials

$$\prod\nolimits_n(\lambda_\nu):=\left\{\sum_{\nu=0}^{n} a_\nu x^{\lambda_\nu}: a_\nu\in\mathbb{R}\right\}. \tag{1}$$

Considering the problem of the approximation of functions $f\in C[0,1]$ by Müntz polynomials, the classical Müntz theorem states (cf. [12]) that $\prod_n(\lambda_\nu)$ is dense in $C[0,1]$ iff $\sum_{\nu=1}^{\infty}(1/\lambda_\nu)=\infty$ and $\lambda_0=0$.

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The question of how fast for functions $f \in C[0, 1]$ the minimal deviation

$$\rho_n(f, (\lambda_\nu), [0, 1]) := \min \left\{ \|f - p_n\|_{[0,1]} : p_n \in \prod_n(\lambda_\nu) \right\} \tag{2}$$

can decrease to zero if n tends to infinity has been studied by many authors (cf., e.g., [2, 4, 5, 8, 13]). In the special case of approximation by usual polynomials the well-known theorem of Jackson (cf. [6]) ensures that the order of approximation will increase with the smoothness of the function being approximated. But the following example shows that such a relation no longer holds in the general case.

For the approximation of $f(x) = x$ by even polynomials, i.e., $\lambda_\nu = 2\nu$, $\nu \in \mathbb{N}$, with a simple transformation we find

$$\rho_n(x, (2\nu), [0, 1]) = \rho_n(\sqrt{x}, (\nu), [0, 1]), \qquad n \in \mathbb{N}.$$

But the minimal deviation $\rho_n(\sqrt{x}, (\nu), [0, 1])$ in approximating $g(x) = \sqrt{x}$ on $[0, 1]$ by usual polynomials is of order $1/n$ (cf. G. Meinardus [11]). Thus we have

$$\rho_n(x, (2\nu), [0, 1]) = O\left(\frac{1}{n}\right) \qquad \text{as} \quad n \to \infty.$$

A better rate of convergence for the approximation by Müntz polynomials from $\prod_n(\lambda_\nu)$ can be expected if we consider the approximation of functions given by means of a series

$$f(z) = \sum_{\nu=0}^{\infty} c_\nu z^{\lambda_\nu}, \qquad c_\nu \in \mathbb{R}.$$

Such series will be called Müntz series. Here in general, z is an element of the Riemann surface of the logarithm denoted by $\mathbb{C}_{\log}$. The approximation of such Müntz series by polynomials from $\prod_n(\lambda_\nu)$ on $[0, 1]$ can be regarded as the generalization of the approximation of holomorphic functions by usual polynomials on a real interval. The latter was treated by Bernstein [3]. In the following we will prove at least qualitatively for the approximation by Müntz polynomials statements analogous to the well-known theorems of Bernstein (cf. Meinardus [11, p. 91f]).

2. Upper Bounds for the Minimal Deviation

In this section we obtain upper bounds for the minimal deviation $\rho_n(f, (\lambda_\nu), [0, 1])$ (cf. (2)) in approximating Müntz series

$$f(z) = \sum_{\nu=0}^{\infty} c_\nu z^{\lambda_\nu}, \qquad c_\nu \in \mathbb{R}, \tag{3}$$

by Müntz polynomials from $\prod_n(\lambda_\nu)$ on the interval $[0, 1]$. Again (λ_ν) is a sequence of positive real numbers λ_ν, $\nu \in \mathbb{N}$,

$$0 \leqslant \lambda_0 < \lambda_1 < \cdots, \qquad \lim_{\nu \to \infty} \lambda_\nu = \infty. \tag{4}$$

We get very general estimates for $\rho_n(f, (\lambda_\nu), [0, 1])$ by assuming that the Müntz series (3) is convergent for a $z = R > 1$. In this case it follows from results on Dirichlet series (cf. [15, p. 355]) that the function f is holomorphic in the domain $\hat{K}_R := \{z \in \mathbb{C}_{\log}: |z| < R\}$.

Theorem 1. *Suppose that with a sequence (λ_ν) satisfying (4) the Müntz series*

$$f(z) = \sum_{\nu=0}^{\infty} c_\nu z^{\lambda_\nu}, \qquad c_\nu \in \mathbb{R},$$

is convergent for a $z = R > 1$. Then with the partial sums s_n, $s_n(x) := \sum_{\nu=0}^{n} c_\nu x^{\lambda_\nu}$, $n \in \mathbb{N}$, and a constant A not depending on n the following inequality holds:

$$\rho_n(f, (\lambda_\nu), [0, 1]) \leqslant \|f - s_n\|_{[0,1]} \leqslant A R^{-\lambda_n}, \qquad n \in \mathbb{N}. \tag{5}$$

Proof. We define

$$B_n := \sum_{\nu=0}^{n} c_\nu R^{\lambda_\nu} := \sum_{\nu=0}^{n} b_\nu, \qquad n \in \mathbb{N},$$

i.e., $b_\nu := c_\nu R^{\lambda_\nu}$, $\nu \in \mathbb{N}$. By assumption it follows that $|B_n| \leqslant B$ for all $n \in \mathbb{N}$ and with the aid of partial Abel summation (cf. [15]) we get

$$\begin{aligned}
\sum_{\nu=n+1}^{\infty} c_\nu x^{\lambda_\nu} &= \sum_{\nu=n+1}^{\infty} b_\nu \left(\frac{x}{R}\right)^{\lambda_\nu} \\
&= \sum_{\nu=n+1}^{\infty} (B_\nu - B_{\nu-1}) \left(\frac{x}{R}\right)^{\lambda_\nu} \\
&= \sum_{\nu=n}^{\infty} B_\nu \left(\left(\frac{x}{R}\right)^{\lambda_\nu} - \left(\frac{x}{R}\right)^{\lambda_{\nu+1}}\right) - B_n \left(\frac{x}{R}\right)^{\lambda_n}
\end{aligned}$$

for all x, $0 \leqslant x < R$. Thus

$$\begin{aligned}\rho_n(f, (\lambda_\nu), [0, 1]) &\leqslant \|f - s_n\|_{[0,1]} \\ &= \max_{x \in [0,1]} \left| \sum_{\nu=n+1}^{\infty} c_\nu x^{\lambda_\nu} \right| \\ &\leqslant B\left(\left(\frac{1}{R}\right)^{\lambda_n} + \sum_{\nu=n}^{\infty} \left(\frac{1}{R}\right)^{\lambda_\nu} - \left(\frac{1}{R}\right)^{\lambda_{\nu+1}}\right) \\ &= 2B\left(\frac{1}{R}\right)^{\lambda_n}\end{aligned}$$

and with $A := 2B$ the assertion is proved.

An inequality similar to (5) is obtained more simply by assuming $\lambda_{\nu+1} - \lambda_\nu \geqslant d > 0$, $\nu \in \mathbb{N}$, for the sequence (λ_ν).

THEOREM 2. *Suppose that with a sequence* (4) *satisfying* $\lambda_{\nu+1} - \lambda_\nu \geqslant d > 0$, $\nu \in \mathbb{N}$, *the Müntz series*

$$f(z) = \sum_{\nu=0}^{\infty} c_\nu z^{\lambda_\nu}, \qquad c_\nu \in \mathbb{R},$$

is convergent for a $z = R > 1$. *Then with the partial sums* $s_n(x) = \sum_{\nu=0}^{n} c_\nu x^{\lambda_\nu}$ *and a constant* A *the inequality*

$$\rho_n(f, (\lambda_\nu), [0, 1]) \leqslant \|f - s_n\|_{[0,1]} \leqslant AR^{-\lambda_{n+1}} \tag{6}$$

holds for all $n \in \mathbb{N}$.

Proof. Since the Müntz series f is convergent for $z = R > 1$ there exists a constant M such that $|c_\nu R^{\lambda_\nu}| \leqslant M$ or $|c_\nu| \leqslant MR^{-\lambda_\nu}$ for all $\nu \in \mathbb{N}$. It now follows that

$$\begin{aligned}\rho_n(f, (\lambda_\nu), [0, 1]) \leqslant \|f - s_n\|_{[0,1]} &= \max_{x \in [0,1]} \left| \sum_{\nu=n+1}^{\infty} c_\nu x^{\lambda_\nu} \right| \\ &\leqslant \sum_{\nu=n+1}^{\infty} |c_\nu| \leqslant M \sum_{\nu=n+1}^{\infty} R^{-\lambda_\nu} \\ &\leqslant MR^{-\lambda_{n+1}} \sum_{\nu=n+1}^{\infty} R^{\lambda_{n+1} - \lambda_\nu} \\ &\leqslant MR^{-\lambda_{n+1}} \sum_{\nu=0}^{\infty} R^{-\nu d} \\ &= M \frac{R^d}{R^d - 1} R^{-\lambda_{n+1}}.\end{aligned}$$

Setting $A := M(R^d/(R^d - 1))$ leads to (6).

3. Bounds for Müntz Polynomials

In order to prove the converse of Theorem 1 or Theorem 2 we need estimates for the absolute value $|p_n(z)|$, $z \in \mathbb{C}_{\log}$, of a Müntz polynomial $p_n(z) = \sum_{v=0}^{n} a_v z^{\lambda_v}$ if only a bound $\|p_n\|_{[0,1]} \leqslant 1$ on the interval $[0, 1]$ is given. Such estimates are obtained by considering the following question asked for the first time by Schwartz in [16].

Problem. How large can the values $|a_k^{(n)}|$, $k = 0(1)\,n$, $n \in \mathbb{N}$, of a polynomial

$$p_n(x) = \sum_{v=0}^{n} a_v^{(n)} x^{\lambda_v} \in \prod_n(\lambda_v)$$

be at most if $\|p_n\|_{[0,1]} \leqslant 1$?

An equivalent formulation is: How large are the values

$$N(k, n; \lambda_v) := \max_{\substack{p_n \in \prod_n(\lambda_v) \\ p_n \neq 0}} \frac{|a_k^{(n)}|}{\|p_n\|_{[0,1]}} \tag{7}$$

for $k \leqslant n$; $k, n \in \mathbb{N}$?

With estimates for the values $N(k, n; \lambda_v)$ we easily find upper bounds for the values $|p_n(z)|$, $z \in \mathbb{C}_{\log}$, of Müntz polynomials p_n.

Lemma 1. *Suppose that the polynomial $p_n \in \prod_n(\lambda_v)$, $n \in \mathbb{N}$,*

$$p_n(x) = \sum_{k=0}^{n} a_k x^{\lambda_k},$$

satisfies $\|p_n\|_{[0,1]} \leqslant P$ with a constant P. Then for all $z \in \mathbb{C}_{\log}$

$$|p_n(z)| \leqslant P \sum_{k=0}^{n} N(k, n; \lambda_v)\, |z|^{\lambda_k}. \tag{8}$$

Proof. Since $\|p_n\|_{[0,1]} \leqslant P$ we have by (7)

$$\frac{|a_k|}{P} \leqslant \frac{|a_k|}{\|p_n\|_{[0,1]}} \leqslant N(k, n; \lambda_v)$$

or

$$|a_k| \leqslant P \cdot N(k, n; \lambda_v), \qquad k = 0(1)\,n.$$

Hence for any $z \in \mathbb{C}_{\log}$

$$|p_n(z)| \leqslant \sum_{k=0}^{n} |a_k| \, |z|^{\lambda_k} \leqslant P \sum_{k=0}^{n} N(k, n; \lambda_v) \, |z|^{\lambda_k}.$$

In the following we seek to obtain estimates for the values $N(k, n; \lambda_v)$. Let $\rho_n^{(k)}(\lambda_v)$, $k \leqslant n$, denote the minimal deviation in approximating the function x^{λ_k} by polynomials from the space $\prod_n(\lambda_v) \backslash \mathrm{span}(x^{\lambda_k})$,

$$\rho_n^{(k)}(\lambda_v) := \min_{\alpha_v} \left\| x^{\lambda_k} - \sum_{\substack{v=0 \\ v \neq k}}^{n} \alpha_v x^{\lambda_v} \right\|_{[0,1]}. \tag{9}$$

There is an interesting connection between the number $N(k, n; \lambda_v)$ and the minimum deviation $\rho_n^{(k)}(\lambda_v)$ stated in:

LEMMA 2. *Let (λ_v) be a fixed sequence* (4). *For all $n \in \mathbb{N}$ the numbers $N(k, n; \lambda_v)$, $\rho_n^{(k)}(\lambda_v)$ (cf.* (7), (9)) *satisfy*

$$N(k, n; \lambda_v) = \frac{1}{\rho_n^{(k)}(\lambda_v)}, \qquad k = 0(1)\, n. \tag{10}$$

Proof. For fixed $n \in \mathbb{N}$ let

$$p_k(x) = \sum_{\substack{v=0 \\ v \neq k}}^{n} a_v^{(k)} x^{\lambda_v}, \qquad k = 0(1)\, n,$$

be polynomials which best approximate the functions x^{λ_k}, i.e., $\rho_n^{(k)}(\lambda_v) = \|x^{\lambda_k} - p_k\|_{[0,1]}$. From definition (7) we have

$$\frac{1}{\rho_n^{(k)}(\lambda_v)} \leqslant N(k, n; \lambda_v), \qquad k = 0(1)\, n. \tag{11}$$

On the other hand for any polynomial $q_n(x) = \sum_{v=0}^{n} b_v^{(n)} x^{\lambda_v} \in \prod_n(\lambda_v)$ it follows that

$$\frac{\|q_n\|_{[0,1]}}{|b_k^{(n)}|} \geqslant \|x^{\lambda_k} - p_k\|_{[0,1]} = \rho_n^{(k)}(\lambda_v)$$

or

$$\frac{1}{\rho_n^{(k)}(\lambda_v)} \geqslant \frac{|b_k^{(n)}|}{\|q_n\|_{[0,1]}}.$$

Consequently

$$\frac{1}{\rho_n^{(k)}(\lambda_\nu)} \geqslant N(k, n; \lambda_\nu)$$

must hold. This together with (11) yields (10).

For the special case of the sequence (λ_ν) with $\lambda_\nu = \nu$, $\nu \in \mathbb{N}$, the numbers $N(k, n; \lambda_\nu)$ have been given by Bernstein (cf. [3, p. 28f]):

$$N(k, n; \lambda_\nu) = 2^{2k} \frac{n(n+k-1)!}{(n-k)!\,(2k)!}.$$

Schwartz has treated the general case with help of functional-analytical methods. He determined in many cases the asymptotic behaviour of the quantity $N(k, n; \lambda_\nu)$ for fixed $k \in \mathbb{N}$, $n \to \infty$.

With the help of Eq. (10) we now obtain estimates for the numbers $N(k, n; \lambda_\nu)$ much more simply:

LEMMA 3. *Let (λ_ν) be a sequence of numbers $0 \leqslant \lambda_0 < \lambda_1 < \cdots$. Then the following inequalities hold for all $k, n \in \mathbb{N}$; $k \leqslant n$:*

$$\frac{1}{\sqrt{2\lambda_k + 1}} \prod_{\substack{\nu=0 \\ \nu \neq k}}^{n} \frac{|\lambda_\nu - \lambda_k|}{\lambda_\nu + \lambda_k + 1} \leqslant \rho_n^{(k)}(\lambda_\nu) \leqslant \prod_{\substack{\nu=0 \\ \nu \neq k}}^{n} \frac{|\lambda_\nu - \lambda_k|}{\lambda_\nu + \lambda_k} \tag{12}$$

resp.

$$\prod_{\substack{\nu=0 \\ \nu \neq k}}^{n} \frac{\lambda_\nu + \lambda_k}{|\lambda_\nu - \lambda_k|} \leqslant N(k, n; \lambda_\nu) \leqslant \sqrt{2\lambda_k + 1} \prod_{\substack{\nu=0 \\ \nu \neq k}}^{n} \frac{\lambda_\nu + \lambda_k + 1}{|\lambda_\nu - \lambda_k|}. \tag{13}$$

Proof. For $f \in C[0, 1]$ let $\|f\|_2$ be the L_2-norm,

$$\|f\|_2 := \left(\int_0^1 f^2(x)\,dx \right)^{1/2}.$$

With arbitrary coefficients $a_\nu \in \mathbb{R}$ we estimate

$$\left\| x^{\lambda_k} - \sum_{\substack{\nu=0 \\ \nu \neq k}}^{n} a_\nu x^{\lambda_\nu} \right\|_2 \leqslant \left\| x^{\lambda_k} - \sum_{\substack{\nu=0 \\ \nu \neq k}}^{n} a_\nu x^{\lambda_\nu} \right\|_{[0,1]}.$$

Choosing numbers a_ν such that

$$\rho_n^{(k)}(\lambda_\nu) = \left\| x^{\lambda_k} - \sum_{\substack{\nu=0 \\ \nu \neq k}}^{n} a_\nu x^{\lambda_\nu} \right\|_{[0,1]}$$

the left side of (12) follows immediately by using the identity (cf. [1, p. 21])

$$\min_{\alpha_\nu} \left\| x^{\lambda_k} - \sum_{\substack{\nu=0\\ \nu\neq k}}^{n} \alpha_\nu x^{\lambda_\nu} \right\|_2 = \frac{1}{\sqrt{2\lambda_k+1}} \prod_{\substack{\nu=0\\ \nu\neq k}}^{n} \frac{|\lambda_\nu - \lambda_k|}{\lambda_\nu + \lambda_k + 1} \tag{14}$$

valid for $\lambda_\nu > -\frac{1}{2}$. In order to get the upper bound for $\rho_n^{(k)}(\lambda_\nu)$ (cf. [5]), with an arbitrary fixed $\alpha > 0$ we put $\beta_\nu = \alpha\lambda_\nu$, $\nu = 0(1)\, n$.

Let us first assume $\lambda_0 > 0$. Then with any numbers $b_\nu \in \mathbb{R}$ the Cauchy–Schwarz inequality yields

$$\begin{aligned} &\left| x^{\beta_k+1/2} - \sum_{\substack{\nu=0\\ \nu\neq k}}^{n} b_\nu x^{\beta_\nu+1/2} \right| \\ &= \left| \left(\beta_k + \frac{1}{2}\right) \int_0^x \left(t^{\beta_k - 1/2} - \sum_{\substack{\nu=0\\ \nu\neq k}}^{n} c_\nu t^{\beta_\nu - 1/2} \right) dt \right| \\ &\leqslant \left(\beta_k + \frac{1}{2}\right) \sqrt{x} \left(\int_0^1 \left(t^{\beta_k - 1/2} - \sum_{\substack{\nu=0\\ \nu\neq k}}^{n} c_\nu t^{\beta_\nu - 1/2} \right)^2 dt \right)^{1/2} \end{aligned} \tag{15}$$

for $x \in [0, 1]$, where we set

$$c_\nu = \frac{b_\nu(\beta_\nu + \frac{1}{2})}{\beta_k + \frac{1}{2}}, \qquad \nu = 0(1)\, n,\ \nu \neq k.$$

Minimizing the right side of (15) by an appropriate choice of numbers c_ν, i.e., b_ν, $\nu = 0(1)\, n$, $\nu \neq k$, we see from (14) that

$$\rho_n^{(k)}(\beta_\nu) \leqslant \left(\beta_k + \frac{1}{2}\right) \frac{1}{\sqrt{2\beta_k}} \prod_{\substack{\nu=0\\ \nu\neq k}}^{n} \frac{|\beta_\nu - \beta_k|}{\beta_\nu + \beta_k}. \tag{16}$$

Since $\beta_\nu = \alpha\lambda_\nu$ we have with any numbers a_ν

$$\left\| \sum_{\nu=0}^{n} a_\nu x^{\beta_\nu} \right\|_{[0,1]} = \left\| \sum_{\nu=0}^{n} a_\nu x^{\lambda_\nu} \right\|_{[0,1]}$$

and therefore $\rho_n^{(k)}(\beta_\nu) = \rho_n^{(k)}(\lambda_\nu)$. Setting $\alpha = 1/2\lambda_k$ we find

$$\rho_n^{(k)}(\lambda_\nu) \leqslant \prod_{\substack{\nu=0\\ \nu\neq k}}^{n} \frac{|\lambda_\nu - \lambda_k|}{\lambda_\nu + \lambda_k}. \tag{17}$$

If $\lambda_0 = 0$ we begin the summation in (15) only with $\nu = 1$ and the above estimates remain valid for $k \in \mathbb{N}$, $0 < k \leqslant n$. If $\lambda_0 = 0$ and $k = 0$, using

$x^{\lambda_\nu}=0$, $\nu=1(1)\,n$, for $x=0$ we get $\rho_n^{(0)}(\lambda_\nu)=1$, $n\in\mathbb{N}$, and (12) is proved. Relation (10) together with (12) leads to (13).

Remark 1. Equality (10) stays true if we consider the problem in the norms

$$\|f\|_p=\left(\int_0^1 |f(x)|^p\,dx\right)^{1/p} \qquad f\in C[0,1],$$

$1\leqslant p<\infty$. Here we are interested in the numbers

$$N_p(k,n;\lambda_\nu):=\max_{\substack{p_n\in\prod_n(\lambda_\nu)\\ p_n\neq 0}} \frac{|a_k^{(n)}|}{\|p_n\|_p}$$

with polynomials $p_n(x)=\sum_{k=0}^n a_k^{(n)}x^{\lambda_k}$ from $\prod_n(\lambda_\nu)$ (cf. (1)). Estimates analogous to those in the proof of Lemma 3 provide inequalities similar to (12) and (13).

For later purposes we need:

LEMMA 4. *Suppose that the numbers* λ_ν, $\nu\in\mathbb{N}$, *of the sequence* (λ_ν) *satisfy*

$$0<d\leqslant\lambda_{\nu+1}-\lambda_\nu\leqslant D<\infty, \qquad \nu\in\mathbb{N}, \tag{18}$$

with constants d, D. *Then for all* $k,n\in\mathbb{N}$, $k\leqslant n$, *we have*

$$\begin{aligned} N(k,n;\lambda_\nu)&\leqslant D(2n+l)^l\left(\frac{D}{d}\right)^n\frac{n^{2k}}{(k!)^2}\\ &\leqslant D(2n+l)^l\left(\frac{D}{d}e^2\right)^n \end{aligned} \tag{19}$$

where l *is the smallest natural number* $l\geqslant(1+2\lambda_0)/D$.

Proof. Let l be the smallest number $l\in\mathbb{N}$ with $l\geqslant(1+\lambda_0 2)/D$; then by (18) we find $\lambda_\nu\leqslant\nu D+\lambda_0$, $\nu\in\mathbb{N}$, $|\lambda_\nu-\lambda_k|\geqslant d\,|\nu-k|$, $k,\nu\in\mathbb{N}$, and consequently

$$\begin{aligned} \frac{\lambda_\nu+\lambda_k+1}{|\lambda_\nu-\lambda_k|}&\leqslant\frac{D(\nu+k)+1+2\lambda_0}{d\,|\nu-k|}=\frac{\nu+k+(1+2\lambda_0)/D}{|\nu-k|}\frac{D}{d}\\ &\leqslant\frac{D}{d}\frac{\nu+k+l}{|\nu-k|}. \end{aligned}$$

With (13) it now follows that

$$N(k, n; \lambda_\nu) \leqslant \sqrt{2\lambda_k+1} \prod_{\substack{\nu=0 \\ \nu \neq k}}^{n} \frac{\lambda_\nu + \lambda_k + 1}{|\lambda_\nu - \lambda_k|}$$

$$\leqslant \frac{\sqrt{2\lambda_k+1}}{2\lambda_k+1} D(2k+l) \prod_{\substack{\nu=0 \\ \nu \neq k}}^{n} \frac{D(\nu+k+l)}{d\,|\nu-k|}$$

$$\leqslant D\left(\frac{D}{d}\right)^n \frac{\prod_{\nu=0}^{n} \nu+k+l}{k!\,(n-k)!}.$$

This using

$$\frac{(n+k+l)\cdots(k+l)}{k!\,(n-k)!} = \frac{(n+k+l)\cdots(n+k)(n+k-1)\cdots(n-k+1)}{(k!)^2(k+1)\cdots(k+l-1)}$$

$$\leqslant (2n+l)^l \frac{n^{2k}}{(k!)^2}$$

yields

$$N(k, n; \lambda_\nu) \leqslant D(2n+l)^l \left(\frac{D}{d}\right)^n \frac{n^{2k}}{(k!)^2}.$$

Observing

$$\frac{n^{2k}}{(k!)^2} \leqslant \frac{n^{2n}}{(n!)^2}$$

for $k \leqslant n$ we obtain with help of the Stirling inequality, $n! > (n/e)^n \sqrt{2\pi n}$, the bound

$$4n\frac{n^{2k}}{(k!)^2} \leqslant 4n\frac{n^{2n}}{(n!)^2} \leqslant 4n\frac{e^{2n}}{2\pi n} \leqslant (e^2)^n, \qquad k \leqslant n,$$

and the assertion is completely established.

The following lemma provides an estimate for $N(k, n; \lambda_\nu)$ (cf. (7)) under the special assumption

$$\lim_{\nu \to \infty} \frac{\nu}{\lambda_\nu} = 0 \tag{20}$$

on the sequence (λ_ν). The approximation by Müntz polynomials with powers x^{λ_ν} where the numbers λ_ν, $\nu \in \mathbb{N}$, satisfy (20) is considered in the next section.

LEMMA 5. *Suppose that the sequence* (λ_ν) *satisfies*

$$\lim_{\nu\to\infty}\frac{\nu}{\lambda_\nu}=0$$
$$\lambda_{\nu+1}-\lambda_\nu\geqslant d>0,\qquad \nu\in\mathbb{N}. \tag{21}$$

Then for any $\varepsilon>0$ *there exists a constant* $A=A(\varepsilon)$ *such that for all* $k, n\in\mathbb{N}$; $k\leqslant n$,

$$N(k,n;\lambda_\nu)\leqslant\sqrt{2\lambda_k+1}\prod_{\substack{\nu=0\\ \nu\neq k}}^{n}\frac{\lambda_\nu+\lambda_k+1}{|\lambda_\nu-\lambda_k|}\leqslant A\cdot e^{\varepsilon\lambda_n}. \tag{22}$$

Proof. We proceed by use of a method which was applied by Levinson (cf. [10]) to determine the growth of certain entire functions. For fixed $k\in\mathbb{N}$, $k\leqslant n$, we split the product

$$\prod_{\substack{\nu=0\\ \nu\neq k}}^{n}\frac{\lambda_\nu+\lambda_k+1}{|\lambda_\nu-\lambda_k|}$$

into two parts, $\prod_1, \prod_2$,

$$\prod_1:=\prod_{\substack{\nu=0\\ \nu\neq k\\ \lambda_\nu\leqslant(3/2)\lambda_k}}^{n}\frac{\lambda_\nu+\lambda_k+1}{|\lambda_\nu-\lambda_k|} \tag{23}$$

$$\prod_2:=\prod_{\substack{\nu=0\\ \lambda_\nu>(3/2)\lambda_k}}^{n}\frac{\lambda_\nu+\lambda_k+1}{\lambda_\nu-\lambda_k}. \tag{24}$$

Let M_k denote the number of powers $\lambda_\nu\neq\lambda_k$, $\nu\in\mathbb{N}$, with $\lambda_\nu\leqslant\frac{3}{2}\lambda_k$. Then M_k has the property:

For any fixed $\delta>0$ there exists a $k_\delta\in\mathbb{N}$ such that

$$M_k\leqslant\lambda_k\delta\qquad\text{for all } k\geqslant k_\delta. \tag{25}$$

Since supposing to the contrary that there exists a sequence of natural numbers k_μ, $\mu\in\mathbb{N}$, with $\lim_{\mu\to\infty}k_\mu=\infty$ and

$$\frac{M_{k_\mu}}{\lambda_{k_\mu}}\geqslant c>0\qquad\text{for all } k_\mu$$

we obtain by $\lambda_{[c\lambda_{k_\mu}]}\leqslant\lambda_{M_{k_\mu}}\leqslant\frac{3}{2}\lambda_{k_\mu}$ that for all sufficiently large k_μ

$$\frac{[c\lambda_{k_\mu}]}{\lambda_{[c\lambda_{k_\mu}]}}\geqslant\frac{2[c\lambda_{k_\mu}]}{3\lambda_{k_\mu}}\geqslant\frac{2(c\lambda_{k_\mu}-1)}{3\lambda_{k_\mu}}=\frac{2}{3}c-\frac{2}{3\lambda_{k_\mu}}\geqslant\frac{1}{3}c,$$

in contradiction to $\lim_{\nu \to \infty}(\nu/\lambda_\nu)=0$. ($[x]$ denotes the largest natural number k, $k \leqslant x$.) Now, for $\lambda_k \geqslant 1$ by $\lambda_\nu \leqslant \frac{3}{2}\lambda_k$ and $|\lambda_\nu - \lambda_k| \geqslant d\,|\nu - k|$ (cf. (21)) we get

$$\begin{aligned}\prod_1 &\leqslant \prod_1 \frac{4\lambda_k}{|\lambda_\nu - \lambda_k|} \\ &\leqslant \left(\frac{4}{d}\right)^{M_k} \lambda_k^{M_k} \prod_1 \frac{1}{|\nu - k|}.\end{aligned} \tag{26}$$

The Stirling inequality $n! > (n/e)^n$, $n \in \mathbb{N}$, and the estimate $\prod_1 |\nu - k| \geqslant ([\frac{1}{2}(M_k - 1)]!)^2$ give

$$\begin{aligned}\prod_1 \frac{1}{|\nu - k|} &\leqslant \frac{e^{2[(1/2)(M_k-1)]}}{[(1/2)(M_k-1)]^{[(1/2)(M_k-1)]\cdot 2}} \\ &\leqslant \frac{e^{M_k}2^{M_k}}{(M_k-2)^{M_k-2}}.\end{aligned}$$

Observing

$$\frac{M_k^{M_k}}{(M_k-2)^{M_k-2}} = M_k^2\left(\frac{M_k}{M_k-2}\right)^{M_k-2} = M_k^2\left(1+\frac{2}{M_k-2}\right)^{M_k-2}$$

and

$$\lim_{k\to\infty} \frac{M_k^2}{c^{M_k}}\left(1+\frac{2}{M_k-2}\right)^{M_k-2} = 0 \qquad \text{for any} \quad c>1$$

we deduce for k sufficiently large

$$\prod_1 \frac{1}{|\nu - k|} \leqslant \frac{e^{2M_k}}{M_k^{M_k}}. \tag{27}$$

Together (26) and (27) lead to

$$\begin{aligned}\prod_1 &\leqslant \left(\frac{4}{d}\right)^{M_k} \frac{\lambda_k^{M_k}e^{2M_k}}{M_k^{M_k}} \\ &= \left(\frac{4}{d}e^2\right)^{M_k} e^{\lambda_k\left(\frac{M_k}{\lambda_k}\log\frac{\lambda_k}{M_k}\right)}.\end{aligned} \tag{28}$$

Using inequality $x\log(1/x) \leqslant \sqrt{x}$ for $x>0$ we have by (25) that

$$\frac{M_k}{\lambda_k}\log\frac{\lambda_k}{M_k} \leqslant \sqrt{\frac{M_k}{\lambda_k}} \leqslant \sqrt{\delta}$$

$$\left(\frac{4}{d}e^2\right)^{M_k} \leqslant e^{\delta\lambda_k \log((4/d)e^2)}.$$

In view of (28), for given $\varepsilon/2 = \sqrt{\delta} + \delta \log((4/d)\, e^2)$ there exists a k_ε such that for $k \geqslant k_\varepsilon$, $k \leqslant n$,

$$\prod_1 \leqslant e^{(\varepsilon/2)\lambda_n} \tag{29}$$

is valid. If $k \leqslant k_\varepsilon$ then the numbers M_k are bounded by a constant $M = M_{k_\varepsilon}$ and since

$$\prod_1 \leqslant \prod_1 \frac{4\lambda_k}{d} \leqslant \left(\frac{4}{d}\lambda_k\right)^M$$
$$= \exp\left(\lambda_n\left(\frac{M}{\lambda_n}\log\frac{4\lambda_k}{d}\right)\right)$$

for $\lambda_k \geqslant 1$ we find for sufficiently large $n \in \mathbb{N}$ in this case inequality (29) as well. Thus we have shown that

$$\prod_1 \leqslant Be^{(\varepsilon/2)\lambda_n} \tag{30}$$

for all $0 \leqslant k \leqslant n$, $n \in \mathbb{N}$, with a constant $B = B(\varepsilon)$.

A similar bound for the other product $\prod_2$ (cf. (24)) is obtained much more simply. By $\lambda_\nu > \frac{3}{2}\lambda_k$ or $\frac{2}{3}\lambda_\nu > \lambda_k$ follows $\lambda_\nu - \lambda_k > \frac{1}{3}\lambda_\nu$ and $\lambda_\nu + \lambda_k + 1 \leqslant 2\lambda_\nu$ for $\lambda_\nu \geqslant 3$. Hence

$$\sqrt{2\lambda_k + 1}\prod_2 \leqslant 3\lambda_n \prod_2 \frac{2\lambda_\nu}{\frac{1}{3}\lambda_\nu} \leqslant 3\lambda_n 6^n$$
$$\leqslant \exp\left(\lambda_n\left(\frac{n\log 6}{\lambda_n} + \frac{\log(3\lambda_n)}{\lambda_n}\right)\right).$$

Since by (21) it follows that $\lim_{n\to\infty}((n\log 6)/\lambda_n) = 0$, for any $\varepsilon/2 > 0$ we can find an $n_\varepsilon \in \mathbb{N}$ such that

$$\sqrt{2\lambda_k + 1}\prod_2 \leqslant e^{(\varepsilon/2)\lambda_n}$$

holds for $n \geq n_\varepsilon$. This together with (30) yields inequality

$$\sqrt{2\lambda_k+1}\prod_{\substack{\nu=0\\ \nu\neq k}}^{n}\frac{\lambda_\nu+\lambda_k+1}{|\lambda_\nu-\lambda_k|}\leq A\cdot e^{\varepsilon\lambda_n}$$

for all $k, n \in \mathbb{N}$, $k \leq n$, with a constant $A = A(\varepsilon)$. In view of (13) the lemma is completely established.

4. Converse Theorems

The estimates for the growth of Müntz polynomials given in the preceding section lead to the following converse theorem.

THEOREM 3. *Suppose that the sequence* (λ_ν) *(cf. (4)) satisfies* $0 < d \leq \lambda_{\nu+1} - \lambda_\nu \leq D < \infty$, $\nu \in \mathbb{N}$, *and that for* $f \in C[0, 1]$ *the bound (cf. (2))*

$$\rho_n(f, (\lambda_\nu), [0, 1]) \leq A \cdot \kappa^{-\lambda_{n+1}}, \qquad n \in \mathbb{N},$$

holds with constants A *and* κ, $\kappa > ((D/d)\, e^2)^{1/d}$. *Then there exists a function* $\hat{f}$ *possessing in* $\mathring{K}_R := \{z \in \mathbb{C}_{\log}: |z| < R\}$ *with* $R := \kappa/((D/d)\, e^2)^{1/d}$ *an absolutely convergent Müntz representation*

$$\hat{f}(z) = \sum_{\nu=0}^{\infty} c_\nu z^{\lambda_\nu}$$

whose restriction to the real interval $[0, 1]$ *coincides with the given function* f.

Proof. By assumption there exist polynomials $q_n \in \prod_n(\lambda_\nu)$ with

$$\|f - q_n\|_{[0,1]} \leq A \cdot \kappa^{-\lambda_{n+1}}, \qquad n \in \mathbb{N}.$$

Setting $p_0 := q_0$, $p_n := q_n - q_{n-1}$, $n \geq 1$, the representation

$$f(x) = q_0(x) + \sum_{n=1}^{\infty} (q_n(x) - q_{n-1}(x)) = \sum_{n=0}^{\infty} p_n(x)$$

is uniformly convergent on $[0, 1]$. Moreover we have

$$\begin{aligned}|p_n(x)| &\leq |q_n(x) - f(x)| + |f(x) - q_{n-1}(x)|\\ &\leq A(\kappa^{-\lambda_{n+1}} + \kappa^{-\lambda_n})\\ &\leq 2A\kappa^{-\lambda_n}, \qquad n \in \mathbb{N}, x \in [0, 1].\end{aligned}$$

Hence for the polynomials $p_n(x)=\sum_{k=0}^{n} a_k^{(n)} x^{\lambda_k} \in \prod_n(\lambda_\nu)$ we get by Lemmas 1 and 4

$$\begin{aligned}|p_n(z)| &\leqslant \sum_{k=0}^{n} |a_k^{(n)}|\, |z|^{\lambda_k} \leqslant \|p_n\|_{[0,1]} \sum_{k=0}^{n} N(k, n; \lambda_\nu)\, |z|^{\lambda_k}\\ &\leqslant 2A\kappa^{-\lambda_n} \sum_{k=0}^{n} D(2n+l)^l \left(\frac{D}{d} e^2\right)^n |z|^{\lambda_k}\\ &\leqslant 2AD(n+1)(2n+l)^l \frac{((D/d)\, e^2)^{\lambda_n/d}}{\kappa^{\lambda_n}} \cdot \begin{cases} 1 & \text{for} \quad |z| \leqslant 1\\ |z|^{\lambda_n} & \text{for} \quad |z| \geqslant 1\end{cases}\end{aligned}$$

for all $z \in \mathbb{C}_{\log}$. Observing the relation

$$\lim_{n\to\infty} (2n+l)^{l+1} \left(\frac{\rho-\varepsilon}{\rho}\right)^n = 0$$

for $\rho > \varepsilon > 0$ it follows with any fixed $\rho, \varepsilon > 0$, $1 < \rho - \varepsilon < \rho < \kappa/((D/d)\, e^2)^{1/d}$ in view of

$$\sum_{n=0}^{\infty} |p_n(z)| \leqslant 2AD \sum_{n=0}^{\infty} (2n+l)^{l+1} \left(\frac{\rho-\varepsilon}{\rho}\right)^{\lambda_n} \left(\rho \frac{((D/d)\, e^2)^{1/d}}{\kappa}\right)^{\lambda_n}$$

for $|z| \leqslant \rho - \varepsilon$ that the series

$$\hat{f}(z) = \sum_{n=0}^{\infty} p_n(z) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_k^{(n)} z^{\lambda_k} \tag{31}$$

is uniformly convergent in the region $K_{\rho-\varepsilon} = \{z \in \mathbb{C}_{\log}: |z| \leqslant \rho - \varepsilon\}$. The expansion (31) of $\hat{f}$ as an absolutely convergent double sum allows a change of summation (cf. [7]) which leads to a Müntz representation

$$\hat{f}(z) = \sum_{k=0}^{\infty} \left(\sum_{n=0}^{\infty} a_k^{(n)}\right) z^{\lambda_k}$$

absolutely convergent in $K_{\rho-\varepsilon}$. The assertion of the theorem now follows by considering the limit $\rho - \varepsilon \to \kappa/((D/d)\, e^2)^{1/d}$.

Remark 2. The previous theorem does not provide the exact converse of Theorem 2. One reason is that the bounds in Theorem 2 obtained with help of the partial sums are not optimal in general. But taking account of the knowledge of the given sequence (λ_ν) we can sharpen the estimates in Section 3 leading to the converse Theorem 3 (cf. (19)) and the bound in

Theorem 2 for the approximation of Müntz series $f(x)=\sum_{\nu=0}^{\infty} c_\nu x^{\lambda_\nu}$ could be improved by evaluating the right side of the inequality

$$\rho_n(f, (\lambda_\nu), [0, 1]) \leqslant \sum_{\mu=n+1}^{\infty} |c_\mu| \, \rho_n(x^{\lambda_\mu}, (\lambda_\nu), [0, 1]).$$

For a certain class of sequences (λ_ν) (cf. (4)) our method yields the exact converse of Theorem 3, namely, if

$$\lim_{\nu\to\infty} \frac{\nu}{\lambda_\nu} = 0, \qquad \lambda_{\nu+1} - \lambda_\nu \geqslant d > 0, \qquad \nu \in \mathbb{N}.$$

THEOREM 4. *Suppose that the sequence* (λ_ν) *satisfies*

$$\lim_{\nu\to\infty} \frac{\nu}{\lambda_\nu} = 0$$

and (32)

$$\lambda_{\nu+1} - \lambda_\nu \geqslant d > 0, \qquad \nu \in \mathbb{N}.$$

If for a function $f \in C[0, 1]$ *the bound*

$$\rho_n(f, (\lambda_\nu), [0, 1]) \leqslant B \cdot R^{-\lambda_{n+1}}, \qquad n \in \mathbb{N},$$

holds with constants B *and* $R > 1$, *then there exists a function* $\hat{f}$ *possessing in* $\mathring{K}_R := \{z \in \mathbb{C}_{\log} : |z| < R\}$ *an absolutely convergent Müntz representation*

$$\hat{f}(z) = \sum_{\nu=0}^{\infty} c_\nu z^{\lambda_\nu}$$

whose restriction to the real interval $[0, 1]$ *coincides with the given function* f.

Proof. The proof parallels the proof of Theorem 3. For the Müntz polynomials $p_n \in \prod_n(\lambda_\nu)$, $n \in \mathbb{N}$, of the expansion

$$f(x) = \sum_{n=0}^{\infty} p_n(x)$$

we again have $\|p_n\|_{[0,1]} \leqslant 2BR^{-\lambda_n}$. By Lemmas 1 and 5 it follows that for any fixed $\varepsilon > 0$

$$\begin{aligned} |p_n(z)| &\leqslant \sum_{k=0}^{n} N(k, n; \lambda_\nu) \, |z|^{\lambda_k} \\ &\leqslant 2BA(\varepsilon)(n+1) \, e^{\varepsilon\lambda_n} \cdot \frac{1}{R^{\lambda_n}} \cdot \begin{cases} 1 & \text{for } |z| \leqslant 1 \\ |z|^{\lambda_n} & \text{for } |z| \geqslant 1 \end{cases} \end{aligned}$$

is valid for all $z \in \mathbb{C}_{\log}$. Let us choose $\rho, \varepsilon > 0$ such that, $\rho > 1$, $1 < \rho e^{\varepsilon} < R$. From inequality

$$\sum_{n=0}^{\infty} |p_n(z)| \leqslant A(\varepsilon)\, 2B \sum_{n=0}^{\infty} (n+1) \left(\frac{e^{\varepsilon}\rho}{R}\right)^{\lambda_n} \qquad \text{for} \quad |z| \leqslant \rho$$

we deduce immediately the uniform convergence of the series

$$\hat{f}(z) = \sum_{n=0}^{\infty} p_n(z) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_k^{(n)} z^{\lambda_k} = \sum_{k=0}^{\infty} \left(\sum_{n=0}^{\infty} a_k^{(n)} \right) z^{\lambda_k} \tag{33}$$

in the domain $K_\rho = \{z \in \mathbb{C}_{\log} : |z| \leqslant \rho\}$. Again the change of summation on the right side of (33) is allowed. Letting $\varepsilon \to 0$ and $\rho \to R$ under the restriction $\rho e^{\varepsilon} < R$ yields the assertion of the theorem.

Suppose that with a sequence satisfying (32) we are given a series $f(z) = \sum_{\nu=0}^{\infty} c_\nu z^{\lambda_\nu}$, $c_\nu \in \mathbb{R}$, convergent for a $z = R > 1$ but divergent for any z with $|z| > R$. Then Theorem 2 assures with the partial sums $s_n(x) = \sum_{\nu=0}^{n} c_\nu x^{\lambda_\nu}$ a rate of convergence

$$\| f - s_n \| \leqslant A R^{-\lambda_{n+1}}, \qquad n \in \mathbb{N}.$$

But by Theorem 4 even with the best approximations $p_n \in \prod_n(\lambda_\nu)$, no bound

$$\rho_n(f, (\lambda_\nu), [0, 1]) = \| f - p_n \|_{[0,1]} \leqslant A \kappa^{-\lambda_{n+1}}, \qquad n \in \mathbb{N},$$

is possible with a constant $\kappa > R$. This means that for such Müntz series the approximation by partial sums essentially provides an optimal rate of convergence.

Remark 3. We note that the condition $\lim_{\nu \to \infty} (\nu/\lambda_\nu) = 0$ in Theorem 4 is different from condition $\sum_{\nu=1}^{\infty} (1/\lambda_\nu) < \infty$ which is characteristic for the density of Müntz polynomials in $C[0, 1]$. The sequence (λ_ν) with $\lambda_\nu = \nu \log(\nu + 1)$, $\nu \in \mathbb{N}$, presents an example of a sequence satisfying $\lim_{\nu \to \infty} (\nu/\lambda_\nu) = 0$ but $\sum_{\nu=1}^{\infty} (1/\lambda_\nu) = \infty$ (cf. [7]).

Remark 4. The statements of Theorems 2 and 4 can be seen in connection with the following generalization of the Fabry gap theorem (cf. [14]):

With a sequence (λ_ν) satisfying $\lim_{\nu \to \infty} (\nu/\lambda_\nu) = 0$, $\lambda_{\nu+1} - \lambda_\nu \geqslant d > 0$, $\nu \in \mathbb{N}$, let a series $f(z) = \sum_{\nu=0}^{\infty} c_\nu z^{\lambda_\nu}$ be given. Then either f is convergent on the whole surface $\mathbb{C}_{\log}$ or the series possesses a bounded "circle" $\{z \in \mathbb{C}_{\log} : |z| = R\}$ of convergence and a holomorphic continuation outside this "circle" does not exist.

5. The Approximation of an Entire Müntz Series

In this section we consider the approximation of functions f allowing a representation

$$f(z)=\sum_{\nu=0}^{\infty} c_\nu z^{\lambda_\nu}, \qquad c_\nu \in \mathbb{R},$$

convergent on the whole surface $\mathbb{C}_{\log}$. We will call such functions an entire Müntz series, and give the following definitions:

(a) An entire Müntz series is said to be of finite order if there exists a σ, $0 \leqslant \sigma < \infty$, such that with constants A, b

$$|f(z)| \leqslant A e^{b|z|^\sigma} \qquad \text{for all } z \in \mathbb{C}_{\log}. \tag{34}$$

(b) The number $\sigma(f)$,

$$\sigma(f) := \inf\{\sigma\colon \sigma \text{ satisfies } (34)\} \tag{35}$$

is called the order (of growth) of the function f.

We generalize the following two theorems concerning the approximation of usual entire functions by usual polynomials. These theorems have been proved by Varga [17] (cf. also Bernstein [3, p. 37]).

Theorem 5. *Let f be an entire function of finite order σ which is real for real values. Then for any $\varepsilon > 0$ there exists a constant $A = A(\varepsilon)$, such that*

$$\rho_n(f, (\nu), [0,1]) \leqslant A \cdot n^{-n/(\sigma+\varepsilon)}, \qquad n \in \mathbb{N}. \tag{36}$$

Theorem 6. *Let $f \in C[0,1]$ be given. Suppose for any $\varepsilon > 0$ there exists a constant $A = A(\varepsilon)$ such that with a σ, $0 \leqslant \sigma < \infty$,* (36) *holds. Then there exists an entire function $\hat{f}$ of order $\sigma(\hat{f}) \leqslant \sigma$, which coincides for real values $x \in [0,1]$ with the given function f.*

The next lemma describes the connection between the order of an entire Müntz series $f(z)=\sum_{\nu=0}^{\infty} a_\nu z^{\lambda_\nu}$ and the asymptotic behaviour of the coefficients a_ν, if $\nu \to \infty$. It generalizes a well-known result for usual entire functions (cf. Levin [9]).

Lemma 6. *Let (λ_ν) be a sequence* (4) *of positive numbers λ_ν, satisfying*

$$\frac{\lambda_\nu}{\log \nu} \geqslant d > 0, \qquad \nu \geqslant 2, \quad \nu \in \mathbb{N}. \tag{37}$$

Suppose f is an entire Müntz series

$$f(z) = \sum_{\nu=0}^{\infty} a_\nu z^{\lambda_\nu}$$

of finite order. Then for the order $\sigma(f)$ of the function f relation

$$\overline{\lim_{\nu\to\infty}} \frac{\lambda_\nu \log \lambda_\nu}{\log(1/|a_\nu|)} = \sigma(f) \tag{38}$$

holds.

Proof. Let σ be the number

$$\overline{\lim_{\nu\to\infty}} \frac{\lambda_\nu \log \lambda_\nu}{\log(1/|a_\nu|)} = \sigma. \tag{39}$$

Then for any $\varepsilon > 0$ there exists a $\nu_\varepsilon \in \mathbb{N}$ such that for all $\nu \geqslant \nu_\varepsilon$, $\nu \in \mathbb{N}$, we have

$$\frac{\lambda_\nu \log \lambda_\nu}{\log(1/|a_\nu|)} \leqslant \sigma + \varepsilon \qquad \text{or} \qquad \frac{1}{|a_\nu|} \geqslant \lambda_\nu^{\lambda_\nu/(\sigma+\varepsilon)}.$$

Consequently with a constant $\gamma = \gamma(\varepsilon)$ for all $\nu \in \mathbb{N}$

$$|a_\nu| \leqslant \gamma \cdot \lambda_\nu^{-\lambda_\nu/(\sigma+\varepsilon)}$$

holds. Hence with any $z = re^{i\varphi} \in \mathbb{C}_{\log}$ the estimate

$$|f(z)| \leqslant \sum_{\nu=0}^{\infty} |a_\nu| \, r^{\lambda_\nu} \leqslant \gamma \cdot \sum_{\nu=0}^{\infty} \frac{r^{\lambda_\nu}}{\lambda_\nu^{\lambda_\nu/(\sigma+\varepsilon)}} \tag{40}$$

is valid. Now we determine the order of the series

$$\sum_{\nu=0}^{\infty} \frac{r^{\lambda_\nu}}{\lambda_\nu^{\alpha\lambda_\nu}} \tag{41}$$

with a number α, $0 < \alpha < \infty$. We set

$$R := [(cr)^{1/\alpha}], \tag{42}$$

where c, $0 < c < \infty$, satisfies

$$\sum_{\nu=0}^{\infty} \frac{1}{c^{\lambda_\nu}} < \infty. \tag{43}$$

($[x]$, $x \in \mathbb{R}$, denotes the largest natural number k, $k \leqslant x$.) Such a number c exists. Indeed, from relation $a^{\log v} = v^{\log a}$; $v, a > 0$; it follows that the series $\sum_{v=1}^{\infty} (1/a^{\log v})$ is convergent for $a > e$. Thus with $c := (2e)^{1/d}$ we find by (37) $(1/c)^{\lambda_v} \leqslant (1/2e)^{\log v}$ and the series (43) is shown to be convergent. Moreover we note

$$R \leqslant (cr)^{1/\alpha} \leqslant R + 1 \tag{44}$$

by (42). Now we write for the series (41)

$$\sum_{v=0}^{\infty} \frac{r^{\lambda_v}}{\lambda_v^{\alpha\lambda_v}} = \sum_{\lambda_v \leqslant R+1} \frac{r^{\lambda_v}}{\lambda_v^{\alpha\lambda_v}} + \sum_{\lambda_v > R+1} \frac{r^{\lambda_v}}{\lambda_v^{\alpha\lambda_v}}. \tag{45}$$

By (44) we have for $\lambda_v > R + 1$

$$\frac{r^{\lambda_v}}{\lambda_v^{\alpha\lambda_v}} \leqslant \left(\frac{r}{(R+1)^\alpha}\right)^{\lambda_v} \leqslant \left(\frac{1}{c}\right)^{\lambda_v}.$$

Thus using (43) it follows that

$$\lim_{R \to \infty} \sum_{\lambda_v > R+1} \frac{r^{\lambda_v}}{\lambda_v^{\alpha\lambda_v}} = 0. \tag{46}$$

Furthermore we obtain for $r \geqslant 1$

$$\begin{aligned} \sum_{\lambda_v \leqslant R+1} \frac{r^{\lambda_v}}{\lambda_v^{\alpha\lambda_v}} &\leqslant r^{(R+1)} \sum_{\lambda_v \leqslant R+1} \frac{1}{\lambda_v^{\alpha\lambda_v}} \\ &\leqslant e^{(R+1)\log r} \sum_{v=0}^{\infty} \frac{1}{\lambda_v^{\alpha\lambda_v}}. \end{aligned} \tag{47}$$

By (37) $\lambda_v^{\alpha\lambda_v} \geqslant (d \log v)^{\alpha\lambda_v}$ and for $v \in \mathbb{N}$ large enough $(d \log v)^{\alpha\lambda_v} > c^{\lambda_v}$. Hence in view of (43) we have shown the convergence

$$\sum_{v=0}^{\infty} \frac{1}{\lambda_v^{\alpha\lambda_v}} \leqslant B.$$

This together with (44) yields the inequality

$$\begin{aligned} \sum_{\lambda_v \leqslant R+1} \frac{r^{\lambda_v}}{\lambda_v^{\alpha\lambda_v}} &\leqslant Be^{R \log r + \log r} \\ &\leqslant Be^{c^{1/\alpha} r^{1/\alpha} \log r + \log r}, \qquad r \geqslant 1. \end{aligned}$$

For any $\varepsilon > 0$ relation $r^{1/\alpha} \log r + \log r \leqslant r^{1/\alpha+\varepsilon}$ holds for large $r > 0$. Thus to any $\varepsilon > 0$ there exists a constant $b = b(\varepsilon)$, such that

$$c^{1/\alpha} r^{1/\alpha} \log r + \log r \leqslant b r^{1/\alpha+\varepsilon}, \qquad r > 0.$$

Applying (46) we conclude from (45) that

$$\sum_{\nu=0}^{\infty} \frac{r^{\lambda_\nu}}{\lambda_\nu^{\alpha\lambda_\nu}} \leqslant A e^{b r^{(1/\alpha+\varepsilon)}}, \qquad r > 0,$$

with a constant $A = A(\varepsilon)$. Setting $1/\alpha = \sigma + \varepsilon$ it follows from (41) and (40)

$$|f(z)| \leqslant \gamma A e^{b|z|^{\sigma+2\varepsilon}}, \qquad z \in \mathbb{C}_{\log},$$

where $\varepsilon > 0$ is arbitrary. Remembering (39) we have established inequality

$$\sigma(f) \leqslant \overline{\lim_{\nu \to \infty}} \frac{\lambda_\nu \log \lambda_\nu}{\log(1/|a_\nu|)} \tag{48}$$

for the order $\sigma(f)$ of the Müntz series f.

Now suppose for any $\varepsilon > 0$ there exist numbers $A = A(\varepsilon)$, $b = b(\varepsilon)$, $A, b > 0$, such that with a σ, $0 \leqslant \sigma < \infty$, relation

$$|f(z)| \leqslant A e^{b|z|^{\sigma+\varepsilon}} \tag{49}$$

holds for all $z \in \mathbb{C}_{\log}$. From Hadamard's formula,

$$a_\nu = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} d(s)\, e^{\lambda_\nu s}\, d\tau, \qquad \operatorname{re} s > \sigma_0,$$

$\nu \in \mathbb{N}$, valid for the Dirichlet series

$$d(s) = \sum_{\nu=0}^{\infty} a_\nu e^{-\lambda_\nu s}, \qquad s = \sigma + i\tau \in \mathbb{C},$$

which are absolutely convergent for $\operatorname{re} s \geqslant \sigma_0$ (cf. [15]) it follows with the transformation $z = e^{-s}$, $z \in \mathbb{C}_{\log}$, for the Müntz series

$$f(z) = \sum_{\nu=0}^{\infty} a_\nu z^{\lambda_\nu} \tag{50}$$

absolutely convergent in $K_R = \{z \in \mathbb{C}_{\log}: |z| \leqslant R\}$, $R > 0$, $z = re^{i\varphi}$, that

$$a_\nu = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} f(z)\, z^{-\lambda_\nu}\, d\varphi, \qquad 0 < |z| < R. \tag{51}$$

Moreover an analogous result for Dirichlet series (cf. [15]) leads to inequality

$$0 \leqslant \log r_c - \log r_a \leqslant \overline{\lim_{v \to \infty}} \frac{\log v}{\lambda_v} \tag{52}$$

for the radius r_c of convergence and the radius r_a of absolute convergence of the series (50). By (37), $\lambda_v/\log v \geqslant d > 0$, $v \in \mathbb{N}$, we get $0 \leqslant \log r_c - \log r_a \leqslant 1/d < \infty$ and consequently the entire Müntz series (50) is absolutely convergent on the whole surface $\mathbb{C}_{\log}$. Using (49) and (51) we find for all $v \in \mathbb{N}$; $r > 0$, $|z| = r$;

$$|a_v| = \left| \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} f(z)\, z^{-\lambda_v}\, d\varphi \right| \leqslant A \frac{e^{br^{\sigma+\varepsilon}}}{r^{\lambda_v}}. \tag{53}$$

Now the function $\varphi_v(r) = e^{br^{\sigma+\varepsilon}}/r^{\lambda_v}$, $r > 0$, takes its absolute minimum at $r = (\lambda_v/b(\sigma+\varepsilon))^{1/(\sigma+\varepsilon)}$. This can be seen from

$$\begin{aligned} \frac{d}{dr} \varphi_v(r) &= \frac{e^{br^{\sigma+\varepsilon}}}{r^{2\lambda_v}} \left(b(\sigma+\varepsilon)\, r^{\sigma+\varepsilon-1+\lambda_v} - \lambda_v r^{\lambda_v - 1}\right) \\ &= \frac{e^{br^{\sigma+\varepsilon}}}{r^{\lambda_v+1}} \left(b(\sigma+\varepsilon)\, r^{\sigma+\varepsilon} - \lambda_v\right). \end{aligned}$$

Thus by (53)

$$|a_v| \leqslant A \frac{e^{\lambda_v/(\sigma+\varepsilon)}}{(\lambda_v/b(\sigma+\varepsilon))^{\lambda_v/(\sigma+\varepsilon)}} = A \left(\frac{eb(\sigma+\varepsilon)}{\lambda_v}\right)^{\lambda_v/(\sigma+\varepsilon)}$$

or

$$\frac{1}{|a_v|} \geqslant \frac{1}{A} \left(\frac{\lambda_v}{eb(\sigma+\varepsilon)}\right)^{\lambda_v/(\sigma+\varepsilon)}$$

and

$$\log \frac{1}{|a_v|} \geqslant \log \frac{1}{A} + \frac{\lambda_v}{\sigma+\varepsilon} \left(\log \lambda_v - \log(eb(\sigma+\varepsilon))\right).$$

Hence we can deduce that for any $\varepsilon > 0$ inequality

$$\log \frac{1}{|a_v|} \geqslant \frac{\lambda_v}{\sigma + 2\varepsilon} \log \lambda_v$$

or

$$\sigma + 2\varepsilon \geq \frac{\lambda_\nu \log \lambda_\nu}{\log(1/|a_\nu|)}$$

holds if ν is chosen sufficiently large. Consequently

$$\sigma \geq \overline{\lim_{\nu \to \infty}} \frac{\lambda_\nu \log \lambda_\nu}{\log(1/|a_\nu|)}$$

and in view of (48) the assertion of the lemma is established. An upper bound for the minimal deviation in approximating an entire Müntz series of finite order by Müntz polynomials is given in:

THEOREM 7. *Let* (λ_ν) *be a sequence* (4) *satisfying*

$$\frac{\lambda_\nu}{\log \nu} \geq d > 0, \qquad \nu \geq 2.$$

Suppose the entire Müntz series

$$f(z) = \sum_{\nu=0}^{\infty} c_\nu z^{\lambda_\nu}, \qquad c_\nu \in \mathbb{R},$$

is of order $\sigma(f) = \sigma$, $0 \leq \sigma < \infty$. *Then for any* $\varepsilon > 0$ *there exists a constant* $A = A(\varepsilon)$, *such that*

$$\rho_n(f, (\lambda_\nu), [0, 1]) \leq \| f - s_n \|_{[0,1]} \leq A \cdot \lambda_n^{-\lambda_n/(\sigma+\varepsilon)}$$

holds with the partial sums $s_n(x) = \sum_{\nu=0}^{n} c_\nu x^{\lambda_\nu}$ *for all* $n \in \mathbb{N}$.

Proof. Since

$$\overline{\lim_{\nu \to \infty}} \frac{\lambda_\nu \log \lambda_\nu}{\log(1/|c_\nu|)} = \sigma$$

by Lemma 6 we have for arbitrary fixed $\varepsilon > 0$

$$\frac{\lambda_\nu \log \lambda_\nu}{\log(1/|c_\nu|)} \leq \sigma + \varepsilon \qquad \text{or} \qquad |c_\nu| \leq \lambda_\nu^{-\lambda_\nu/(\sigma+\varepsilon)}$$

if ν is large enough. Hence there exists a $\gamma = \gamma(\varepsilon) > 0$, such that

$$|c_\nu| \leq \gamma \lambda_\nu^{-\lambda_\nu/(\sigma+\varepsilon)}, \qquad \nu \in \mathbb{N}.$$

Thus we get with the partial sums s_n of the series f

$$\begin{aligned}\|f-s_n\|_{[0,1]} &= \max_{x\in[0,1]}\left|\sum_{v=n+1}^{\infty} c_v x^{\lambda_v}\right| \\ &\leqslant \gamma \sum_{v=n+1}^{\infty} \lambda_v^{-\lambda_v/(\sigma+\varepsilon)}.\end{aligned} \tag{54}$$

In the same way as in the proof of Lemma 6 we can show the existence of a constant c, $0<c<\infty$, satisfying

$$\sum_{v=0}^{\infty}\left(\frac{1}{c}\right)^{\lambda_v/(\sigma+\varepsilon)}<\infty.$$

Consequently the numbers

$$B_n := \sum_{v=0}^{n}\left(\frac{1}{c}\right)^{\lambda_v/(\sigma+\varepsilon)}$$

are bounded,

$$B_n \leqslant B, \qquad n\in\mathbb{N}. \tag{55}$$

By Abel's method of summation by parts we find using (55)

$$\begin{aligned}\sum_{v=n+1}^{\infty}\left(\frac{1}{\lambda_v}\right)^{\lambda_v/(\sigma+\varepsilon)} &= \sum_{v=n+1}^{\infty}\left(\frac{c}{\lambda_v}\right)^{\lambda_v/(\sigma+\varepsilon)}\left(\frac{1}{c}\right)^{\lambda_v/(\sigma+\varepsilon)} \\ &= \sum_{v=n+1}^{\infty}\left(\frac{c}{\lambda_v}\right)^{\lambda_v/(\sigma+\varepsilon)}(B_v-B_{v-1}) \\ &= \sum_{v=n}^{\infty} B_v\left(\left(\frac{c}{\lambda_v}\right)^{\lambda_v/(\sigma+\varepsilon)}-\left(\frac{c}{\lambda_{v+1}}\right)^{\lambda_{v+1}/(\sigma+\varepsilon)}\right) \\ &\quad - B_n\left(\frac{c}{\lambda_n}\right)^{\lambda_n/(\sigma+\varepsilon)} \\ &\leqslant B\left(\left(\frac{c}{\lambda_n}\right)^{\lambda_n/(\sigma+\varepsilon)}+\sum_{v=n}^{\infty}\left(\left(\frac{c}{\lambda_v}\right)^{\lambda_v/(\sigma+\varepsilon)}\right.\right. \\ &\quad \left.\left.-\left(\frac{c}{\lambda_{v+1}}\right)^{\lambda_{v+1}/(\sigma+\varepsilon)}\right)\right) \\ &\leqslant 2B\left(\frac{c}{\lambda_n}\right)^{\lambda_n/(\sigma+\varepsilon)} \qquad \text{for} \quad c<\lambda_n.\end{aligned}$$

Taking account of the fact that for any $\varepsilon > 0$ and fixed $c > 0$

$$\left(\frac{c}{\lambda_n}\right)^{\lambda_n/(\sigma+\varepsilon)} \leqslant \left(\frac{1}{\lambda_n}\right)^{\lambda_n/(\sigma+2\varepsilon)}$$

holds if $n \in \mathbb{N}$ is large enough it follows by (54) that

$$\begin{aligned}\rho_n(f, (\lambda_\nu), [0, 1]) &\leqslant \|f - s_n\|_{[0,1]}\\ &\leqslant A\lambda_n^{-\lambda_n/(\sigma+2\varepsilon)}\end{aligned}$$

for all $n \in \mathbb{N}$ with a constant $A = A(\varepsilon)$.

Under stronger hypotheses on the sequence (λ_ν) we obtain the converse in:

THEOREM 8. *Suppose that the sequence (λ_ν) satisfies $0 < d \leqslant \lambda_{\nu+1} - \lambda_\nu \leqslant D < \infty$, $\nu \in \mathbb{N}$. Let $f \in C[0, 1]$ be given. Suppose for any $\varepsilon > 0$ there exists a constant $A = A(\varepsilon)$ such that with a σ, $0 \leqslant \sigma < \infty$, we have*

$$\rho_n(f, (\lambda_\nu), [0, 1]) \leqslant A \cdot \lambda_n^{-\lambda_n/(\sigma+\varepsilon)}, \qquad n \in \mathbb{N}.$$

Then there exists an entire Müntz series $\hat{f}$,

$$\hat{f}(z) = \sum_{\nu=0}^{\infty} c_\nu z^{\lambda_\nu},$$

of order $\sigma(\hat{f})$, $0 \leqslant \sigma(\hat{f}) \leqslant \sigma$, absolutely convergent on the whole surface $\mathbb{C}_{\log}$ which coincides for real $x \in [0, 1]$ with the given function f.

Proof. We use the same notations as in the proof of Theorem 3. We consider the expansion

$$f(x) = \sum_{n=0}^{\infty} p_n(x)$$

uniformly convergent for $x \in [0, 1]$ with polynomials $p_n \in \prod_n(\lambda_\nu)$, $p_n(x) = \sum_{\nu=0}^{n} a_\nu^{(n)} x^{\lambda_\nu}$, satisfying

$$\begin{aligned}\|p_n\|_{[0,1]} &\leqslant A\left(\left(\frac{1}{\lambda_n}\right)^{\lambda_n/(\sigma+\varepsilon)} + \left(\frac{1}{\lambda_{n-1}}\right)^{\lambda_{n-1}/(\sigma+\varepsilon)}\right)\\ &\leqslant 2A\left(\frac{1}{\lambda_{n-1}}\right)^{\lambda_{n-1}/(\sigma+\varepsilon)} \qquad n \geqslant 1.\end{aligned}$$

By Lemmas 1 and 4 it follows for all $z \in \mathbb{C}_{\log}$, $n \geqslant 1$, with $l := [(1+2\lambda_0)/D+1]$ that

$$|p_n(z)| \leqslant 2A \left(\frac{1}{\lambda_{n-1}}\right)^{\lambda_{n-1}/(\sigma+\varepsilon)} \sum_{\nu=0}^{n} (2n+l)^l D \left(\frac{D}{d} e^2\right)^n |z|^{\lambda_\nu}.$$

Thus setting $k := ((D/d)\, e^2)^{1/d}$ the bound

$$|p_n(z)| \leqslant 2AD(2n+l)^{l+1} r^{\lambda_n} k^{\lambda_n} \left(\frac{1}{\lambda_{n-1}}\right)^{\lambda_{n-1}/(\sigma+\varepsilon)} \qquad n \geqslant 1, \tag{56}$$

is valid for $|z| \leqslant r$, $r \geqslant 1$. Furthermore we get for any fixed $\varepsilon > 0$

$$(2n+l)^{l+1} k^{\lambda_n} \left(\frac{1}{\lambda_{n-1}}\right)^{\lambda_{n-1}/(\sigma+\varepsilon)} \leqslant \left(\frac{1}{\lambda_n}\right)^{\lambda_n/(\sigma+2\varepsilon)} \tag{57}$$

if $n \in \mathbb{N}$ is large enough. Together (56) and (57) ensure the existence of a constant $E = E(\varepsilon)$ such that

$$\sum_{n=0}^{\infty} |p_n(z)| \leqslant E\left(1 + \sum_{n=0}^{\infty} \left(\frac{1}{\lambda_n}\right)^{\lambda_n/(\sigma+2\varepsilon)} r^{\lambda_n}\right) \tag{58}$$

for all $|z| \leqslant r$, $r > 0$. Setting

$$a_n = \left(\frac{1}{\lambda_n}\right)^{\lambda_n/(\sigma+2\varepsilon)} \qquad n \in \mathbb{N},$$

it follows from Lemma 6 that the sum on the right side of (58) is of order

$$\overline{\lim_{n\to\infty}} \frac{\lambda_n \log \lambda_n}{\log(1/|a_n|)} = \sigma + 2\varepsilon,$$

where $\varepsilon > 0$ is arbitrary. Applying a change of summation

$$\hat{f}(z) = \sum_{n=0}^{\infty} p_n(z) = \sum_{k=0}^{\infty} \left(\sum_{n=0}^{\infty} a_k^{(n)}\right) z^{\lambda_k}$$

we have shown $\hat{f}$ to be a Müntz series of order $\sigma(\hat{f})$, $0 \leqslant \sigma(\hat{f}) \leqslant \sigma$, absolutely convergent on the whole surface $\mathbb{C}_{\log}$.

Remark 5. Remembering the fact mentioned in Remark 1 that formulas analogous to (8) and (13) hold for the problem in the L_p-norm, $1 \leqslant p < \infty$, simple modifications in the proofs of the preceding theorems show that statements analogous to all theorems in Section 4 and 5 are valid in the L_p-norms, $1 \leqslant p < \infty$.

Furthermore, simple transformation arguments lead to analogous results for the problem of the approximation of functions by Müntz polynomials in the interval $[0, b]$ for $b > 0$.

6. The Approximation of Dirichlet Series by Dirichlet Polynomials

With the aid of the transformation

$$x = e^{-\sigma} \qquad \text{resp.} \qquad \sigma = -\log x$$

we obtain one to one correspondence between functions $f(x) \in C[0, 1]$ and functions $F(\sigma) \in C[0, \infty]$ by setting

$$f(e^{-\sigma}) = F(\sigma) \qquad \text{resp.} \qquad F(-\log x) = f(x).$$

Moreover

$$\| f(x) \|_{[0,1]} = \| F(\sigma) \|_{[0,\infty]}. \tag{59}$$

With

$$z = e^{-s}, \qquad s = \sigma + i\tau \in \mathbb{C},$$

$$s = -\log |z| - i\varphi, \qquad z = |z|\, e^{i\varphi} \in \mathbb{C}_{\log},$$

by

$$f(e^{-s}) = F(s), \qquad F(-\log z) = f(z) \tag{60}$$

to any function $f(z)$ which is holomorphic in a domain $G \in \mathbb{C}_{\log}$ we have assigned a function $F(s)$ which is holomorphic in the corresponding domain of the $\mathbb{C}$ plane and vice versa. For instance, the functions

$$f(z) = z^{\lambda}, \qquad F(s) = e^{-\lambda s}, \qquad \lambda \in \mathbb{C},$$

are corresponding under the transformation (60). Hence, all assertions in the preceding sections concerning the approximation of Müntz series

$$f(z) = \sum_{\nu=0}^{\infty} c_\nu z^{\lambda_\nu}$$

by Müntz polynomials from $\prod_n(\lambda_\nu)$ on $[0, 1]$ can be modified into corresponding statements concerning the approximation of Dirichlet series

$$d(s) = \sum_{\nu=0}^{\infty} c_\nu e^{-\lambda_\nu s}$$

by Dirichlet polynomials from $\Delta_n(\lambda_\nu)$,

$$\Delta_n(\lambda_\nu) = \left\{ \sum_{\nu=0}^{n} a_\nu e^{-\lambda_\nu \sigma} : a_\nu \in \mathbb{R} \right\},$$

on the interval $[0, \infty]$. Hereby we have only to take into consideration that under the transformation $s = -\log z$ the domain $K_R \subset \mathbb{C}_{\log}$, $R > 0$, $K_R = \{z \in \mathbb{C}_{\log} : |z| \leqslant R\}$, corresponds to the region $H_R \subset \mathbb{C}$,

$$H_R = \{s = \sigma + i\tau \in \mathbb{C} : \sigma \geqslant -\log R\}.$$

Similarly assertions analogous to those of Section 3 are valid for Dirichlet polynomials

$$d_n(\sigma) = \sum_{\nu=0}^{n} a_\nu^{(n)} e^{-\lambda_\nu \sigma}.$$

Defining

$$\hat{N}(k, n; \lambda_\nu) = \max_{\substack{d_n \in \Delta_n(\lambda_\nu) \\ d_n \not\equiv 0}} \frac{|a_k^{(n)}|}{\|d_n\|_{[0,\infty]}}, \qquad k \leqslant n;\, k, n \in \mathbb{N},$$

it follows by (59) that $\|p_n\|_{[0,1]} = \|d_n\|_{[0,\infty]}$ with $p_n(x) = \sum_{\nu=0}^{n} a_\nu^{(n)} x^{\lambda_\nu}$ and consequently (cf. (7))

$$\hat{N}(k, n; \lambda_\nu) = N(k, n; \lambda_\nu).$$

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